The Semantic View of Scientific Theories Meets Correspondence Theory in a First-Order Pragmatist Semantics

F. Thomas Burke
Department of Philosophy, University of South Carolina, 401 Byrnes Building, Columbia, South Carolina 29208, USA. [E-mail address: burket@mailbox.sc.edu]

Abstract
Instead of interpreting predicate symbols in a first-order language $L_1$ by their extensions (as sets of tuples of individuals) in a given domain of discourse, we show how to interpret them using classes of models for a language $L_0$ that treats each ordered tuple of objects in the domain of discourse of $L_1$ as a world. This is essentially an extrapolation of the semantic conception of scientific theories to ordinary first-order predicate symbols. It also reverses the orientation of correspondence theory which shows how to characterize various modal sentential logics by way of specific axioms in respective first-order correspondence languages. This work is part of a project aimed at evaluating the merits of C. S. Peirce’s operationalist conception of meaning as formulated (vaguely) in the so-called pragmatic maxim, namely, a methodological “maxim of logic” addressed to the issue of how best to define one’s terms clearly.

Keywords: Correspondence theory, Semantic view of scientific theories, Extensionalism, Pragmatism, Operationalism, Inferentialism, Material inference, Interpretation, Semantics

1. Introduction
To put things into perspective, this article is a small part of a project aimed at assessing the merits of classical American pragmatism as a source of insights concerning the nature of linguistic meaning. One might have a number of reasons for doing that, but one significant payoff is a demystification if not a systematic account of material inference [1–5]. This project bears directly on the foundations of mathematics insofar as it concerns the characterization of fundamental mathematical concepts, including those that in
turn have been essential to the development of formal semantics. This larger project is focused around two basic ideas, as follows:

(1) Analytic pragmatism comes in two complimentary versions: *operationalist pragmatism*, following Peirce [6–17], and *inferentialist pragmatism*, following James [18–24].

(2) Brandom [4, 25, 26] has developed a version of inferentialist pragmatism, including a distinctive account of material inference. An operationalist pragmatism, on the other hand, may be cashed out as follows: classes of *models* for ability-based dynamic languages (ABDLs) can be used in place of sets of *things* to interpret the predicate symbols in a first-order language. The notion of material inference will be cashed out in terms of relationships among such classes of models.

An *ability-based* dynamic logic is a kind of dynamic logic that takes designated agent-oriented *abilities to act* and respective *registerable results of executing such abilities* as its starting point such that the standard elements of a dynamic logic (variables, programs, etc.) are secondary constructions rather than primitive. Burke [27, 28] shows that the proofs of soundness and completeness for a particular infinitary proof system in Goldblatt [29] carry over to ABDLs. An ABDL accommodates operational pragmatism insofar as it takes actions and their results as its starting point—as the basis in terms of which one's fundamental terms are to be defined.

These latter details, while key to a pragmatist take on logic, are not crucial here. Rather, to step back and gain a broader perspective, there are two related conjectures that are the present focus:

(3) The notion that classes of models for one language may be used to interpret predicate symbols of another language has been exploited independently of the notion that ability-based dynamic logics may be used to formalize an operationalist reading of the pragmatic maxim. So-called correspondence theory in philosophical logic [30–32], though limited in scope, is one case in point. The method of using classes of models of one language to interpret another language is in any case a technique that may be used generally, independently of pragmatism. The models do not have to be ABDL models.

(4) The simple idea of replacing sets of objects with classes of models is a general technique that, in principle, might be iterated indefinitely in two directions. Namely, (a) the constituent predicate symbols of a given target language $L_1$ may be interpreted as classes of models for some source language $L_0$, while at the same time (b) models for $L_1$, as a source language, may serve to interpret the predicate symbols in some other target language $L_2$.

To better understand idea (2), we will focus here on illustrating a simple example of idea (3).
2. Details

A truth-value assignment to all of the atomic sentences in a language $L_0$ of sentential logic is a simple example of a model for a language. Classes of such models, for properly designed languages $L_0$, may be used to interpret predicate symbols in a broad range of (if not all) first-order languages. As a concrete example, we will show that this is the case for the blocks language $L_{bl}$ used in [33] to teach elementary first-order logic (see Appendix A below). One upshot will be an account of “analytic consequence” (and thus material inference) as illustrated so nicely by this particular language.

2.1. Correspondence theory

Idea (2)—the idea that predicate symbols in a given first-order language may be interpreted by “classes of models” of a certain kind (namely, models for appropriate operation-based languages)—was originally motivated (for this author) by the semantic conception of scientific theories [34–42]. The semantic conception of scientific theories is of course concerned with just that—scientific theories. The idea here is to borrow the “class of models” idea as the basis for developing a formal alternative to standard extensional semantics for first-order languages—an alternative that in particular reflects Peirce’s operationalist conception of meaning.

Meanwhile, consider the following passage by Blackburn and van Benthem [32] where PROP is a set $\{p, q, \ldots\}$ of atomic sentence symbols and MOD is a set $\{m, m', m'', \ldots\}$ of modality symbols:

So what is a Kripke model? No mystery here. A Kripke model $(W, \{R^m\}_{m \in \text{MOD}}, V)$ is what model theorists call a relational structure. That is, we have a domain of quantification $W$, a collection of binary relations over this domain, and a collection of unary relations as well (after all, $V(p)$ is a unary relation for each $p \in \text{PROP}$). But this means that we are not forced to talk about Kripke models using modal languages: they provide us with everything needed to interpret classical languages too. For example, to talk about a model $(W, \{R^m\}_{m \in \text{MOD}}, V)$ using first-order logic we would simply make use of a first-order language with a binary relation symbol $R^m$ for every $m \in \text{MOD}$, and a unary relation symbol $P$ for every $p \in \text{PROP}$. Modal logicians have a name for this language: they call it the first-order correspondence language (for the basic modal language over PROP and MOD). [p. 10]

There is nothing surprising here; but this statement makes it clear that Kripke models can be used to interpret certain first-order languages exten-
sionally. This illustrates idea (3) at least in a limited way, though it is interesting to contrast this reversal of orientation with the more standard way of talking about such correspondences, namely, where the first-order correspondence language is more naturally regarded as interpreting the modal language [30, 43]. Given an arbitrary first-order language, one does not usually look for a modal sentential language for which the former is its first-order correspondence language.

Yet, that is basically what we want to be able to do in order to interpret an arbitrary first-order language using classes of models of some appropriate kind to interpret its predicate symbols. Correspondence theory, as a systematic study of the first-order definability of modal formulas, may be more useful in explaining what this even means beyond occasional analogical references to the semantic conception of scientific theories.

Here is a simple illustration. If we add a single standard box operator to a sentential language $L_0$, the set of all truth-value assignments to all of the atomic sentences in $L_0$ determines an $S5$ truth-valuation model $\mathcal{M} = \langle W, R, \mathcal{V} \rangle$ for the respective modal sentential language $L_{0\Box}$ resulting from the stipulation that

- $W$ is the set of all truth assignments $r$ to the atomic sentences in $L_0$,
- the accessibility relation $R$ is universal; namely, for all $i, j$, row $r_i$ is accessible from row $r_j$, i.e., $\langle r_i, r_j \rangle \in R$, and
- $\mathcal{V}(p) = \{ r \in W | r(p) = \text{TRUE} \}$ for each atomic sentence $p$ in $L_0$.

This essentially goes no further than Carnap’s original semantics for a modal logic in terms of “state descriptions” (as worlds) containing sets of atomic propositions, where the set of valid formulas in such a model is just the set of theorems in Lewis’s $S5$ logic [44–47].

Correspondence theory for modal sentential logic [30–32] says then that there is a structure $S_{2\Box}$ (that is, $\mathcal{M}$ itself) for a respective first-order correspondence language $L_{0\Box}$ where atomic sentences $p$ in $L_{0\Box}$ are mapped to unary predicate symbols $P$ in $L_{0\Box}$ (or more precisely, for each variable $x$, to unary atomic formulas $P(x)$), the accessibility relation $R$ is mapped to a universal binary relation symbol $R$, and

$\mathcal{M}, r \models \varphi$ iff $S_{2\Box} \models \varphi^*_{x} [x \leftarrow r]$ 

given that

$P^*_{x} = P(x)$,
$\bot^*_{x} = \bot$,
$(\varphi \rightarrow \psi)^*_{x} = \varphi^*_{x} \rightarrow \psi^*_{x}$,
$(\Box \varphi)^*_{x} = \forall x (R(x, y) \rightarrow \varphi^*_{y}) = \forall y (\varphi^*_{y})$ for some $y \neq x$. 
The notation ‘\([x \leftarrow r]\)’ means “assign object \(r\) to the free variable \(x\).” The point here is that \(V\) (as part of the model \(M\) for \(L_0\)) can be said to provide an interpretation of \(L_0^*\)-predicate symbols in the sense of idea (3)—in the sense, namely, that for any predicate symbol \(P\) in the target language \(L_0^*\) there is a unique proposition symbol \(p\) in the source language \(L_0\) where \(p^*_x = P(x)\) and where the set of truth-valuations \(V(p) = \{ r \in W \mid r(p) = \text{true} \}\) (as the “proposition” expressed by \(p\) in \(L_0\)) can be said to be the \(L_0^*\)-extension of predicate symbol \(P\) in \(L_0^*\). This relationship between \(P\) and \(V(p)\) holds because \(L_0^*\) will have been defined that way: the target language \(L_0^*\) will have been constructed only after being given the source language \(L_0\).

The point here is that, in contrast with the model \(M\), the set \(\{ r \in W \mid r(p) = \text{true} \}\) (as the \(L_0^*\)-extension of a corresponding predicate symbol \(P\) in the target language \(L_0^*\)) is essentially a set of “state descriptions” and thus a set of truth-functional “models” in its own right—models, namely, for the ordinary non-modal sentential source language \(L_0\) that we started with.

In the present context, we want to go in the reverse direction. We are already given a “typical” target first-order language—the blocks language \(L_{tpl}\)—and we want to find a source language \(L_0\) such that sets of \(L_0\)-models yield suitable \(L_0\)-extensions for predicate symbols \(P\) in \(L_{tpl}\). To say the least, it is not obvious that the latter blocks language is a first-order correspondence language for any modal sentential language \(L_0\) in the above sense; though if it were, we would have a familiar illustration of idea (3).

One obvious issue is that a first-order correspondence language \(L^*\) for a given modal sentential logic \(L\) has only unary predicate symbols (one for each atomic sentence in \(L\)) and binary predicate symbols (one for each modal operator in \(L\)). The blocks language is not like that. We might eliminate the one ternary predicate symbol in \(L_{tpl}\) to fit this particular requirement, but this kind of \(ad\ hoc\) move is not acceptable. If it helps to achieve anything at all, it nevertheless puts off the question of how to interpret first-order languages other than those with \(n\)-ary predicate symbols only for \(n \leq 2\).

Another issue is that the first-order variables in \(L^*\) range over individuals that in \(L\) are treated like worlds. The latter notion has always (for the most part) been merely heuristic, but it is instructive here. If we wanted to view \(L_{tpl}\) as a first-order correspondence language for some modal sentential language \(L_0\), then individual blocks in models for \(L_{tpl}\) would be treated like worlds in models for \(L_0\).

It is not clear that we can do this directly. To make this work, it is proposed that we proceed in a number of steps. We will specify (in enough detail) the four languages listed in Table 1 and show that the three depicted “correspondences” hold between consecutive pairs. The result will be that the predicate symbols in \(L_{tpl}\) can be interpreted using classes of \(L_0\) models.
Table 1: Four languages

\[
\begin{array}{c|c|c|c|c}
L_{tpl} & \overset{1}{\leftrightarrow} & L_{0\Box}^* & \overset{2}{\leftrightarrow} & L_{0\Box} & \overset{3}{\leftrightarrow} & L_0
\end{array}
\]

(and that analytic consequence and material inference relative to models for \(L_{tpl}\) can be explained in terms of relationships among \(L_0\) models).

Here first is a brief overview. Given the blocks language \(L_{tpl}\), we will first define a semantically equivalent monadic first-order language \(L_{0\Box}^*\) where each predicate symbol in \(L_{tpl}\) corresponds to a unique and distinct unary predicate in \(L_{0\Box}^*\). In fact, we may use the same predicate symbols in the two languages so that the mapping between the two sets of predicate symbols is readily obvious—though it must be remembered that the binary and ternary predicate symbols in \(L_{tpl}\) will be reinterpreted as unary predicate symbols in \(L_{0\Box}^*\). The correspondence relation \(\overset{1}{\leftrightarrow}\) (if we may call it that) is one of semantic isomorphism.

The scripting notation should become clear as we proceed. In particular, the asterisk suggests that \(L_{0\Box}^*\) will be something like a standard first-order correspondence language for a particular modal sentential language \(L_{0\Box}\). The correspondence relation \(\overset{2}{\leftrightarrow}\) means just that, namely, that \(L_{0\Box}^*\) is a perhaps non-standard first-order correspondence language for \(L_{0\Box}\). Roughly, each atomic sentence symbol \(p\) in \(L_{0\Box}\) will correspond uniquely to a unary predicate symbol \(P\) in \(L_{0\Box}^*\) and thereby to the respective predicate symbol (not necessarily unary) in the blocks language \(L_{tpl}\).

The language \(L_0\), then, is the simpler sentential language obtained as a fragment of \(L_{0\Box}\) by eliminating the modal operator. The modal sentential language \(L_{0\Box}\) (not limited to \(S5\) models) is richer than \(L_0\) in significant ways; but we are interested here only in particular \(S5\) models \(\mathcal{M}_w\). Relative to a fixed blocks world \(w\), we can build an appropriate \(S5\) model \(\mathcal{M}_w = (\mathcal{W}_w, \mathcal{R}_w, \mathcal{V}_w)\) for the modal sentential language \(L_{0\Box}\) such that the world-assignment \(\mathcal{V}_w(p)\) will serve as an extensional \(L_{0\Box}\)-interpretation of \(p\) in \(L_{0\Box}^*\). This set of \(\mathcal{W}_w\)-worlds will serve in turn as an extensional \(L_0\)-interpretation of the same predicate symbol \(P\) (e.g., \(\text{Cube}\)) in \(L_{tpl}\) relative to the blocks world \(w\) insofar as each \(\mathcal{W}_w\)-world corresponds to a truth-value assignment to atomic sentences in \(L_0\). The correspondence relation \(\overset{3}{\leftrightarrow}\) means essentially (intuitively, loosely) that \(\mathcal{M}_w\) is equivalent to a composite truth-table \(\mathcal{T}_0\) for all of the sentences in the language of \(L_0\) where the “reference columns” of \(\mathcal{T}_0\) include all of the atomic sentences of \(L_0\). “Worlds” in \(\mathcal{M}_w\) will thus correspond to rows in \(\mathcal{T}_0\).

In greater detail, we will now work left-to-right through Table 1 one step
at a time.

2.2. The first-order blocks language $L_{\text{tpl}}$

The predicate symbols $P$ of the blocks language and their informal meanings are spelled out in Appendix A and summarized in Tables A.2 and A.3 (pages 34–35). The language has six names $c$, six unary predicate symbols $P^1$, twelve binary predicate symbols $P^2$ (including identity), one ternary predicate symbol $P^3$, variables $x$, standard sentential connectives, and the quantifier symbols $\forall, \exists$. For the sake of brevity, we will work with the following simplified substitute for the full-blown blocks language:

$$t ::= c \mid x$$

$$\varphi ::= P^1(t) \mid P^2(t_1, t_2) \mid P^3(t_1, t_2, t_3) \mid \bot \mid \varphi_1 \rightarrow \varphi_2 \mid \forall x \varphi$$

Each extensional structure $\mathcal{S}_w$ for $L_{\text{tpl}}$ is defined relative to a specific blocks world $w$—essentially a standard-sized checkerboard populated by some number of small, medium, and/or large tetrahedra, cubes, and/or dodecahedra. Such worlds can have at most twelve blocks, at most six of which may be named. There are only three possible shapes and three possible sizes of blocks, and blocks can be located at any of only sixty-four possible positions on the board (see Appendix B). Various peculiar features of some of the intended meanings of the predicate symbols are given in Table A.3. A structure $\mathcal{S}_w$ will be defined so as to specify referents for each of the names and extensions for each of the nineteen predicate symbols. Let $U_w$ denote the set of all blocks in world $w$. Then, as one would expect,

- $\mathcal{S}_w(\forall) = U_w$
- $\mathcal{S}_w(c) \in U_w$ for each name $c$
- $\mathcal{S}_w(P^1) \subseteq U_w$ for each of the six unary predicate symbols $P^1$
- $\mathcal{S}_w(P^2) \subseteq U_w^2$ for each of the twelve binary predicate symbols $P^2$
- $\mathcal{S}_w(P^3) \subseteq U_w^3$ for the one ternary predicate symbol $P^3$

An $\mathcal{S}_w$-valuation, then, is a total or partial (possibly empty) function

$$V_w : \text{Variables}_{\text{tpl}} \rightarrow U_w$$

assigning a block $V_w(x)$ in $U_w$ to each $L_{\text{tpl}}$-variable $x$ in its domain. Let $V_w\emptyset$ denote the empty $\mathcal{S}_w$-valuation. The function $V_w$ extends uniquely to $\text{Terms}_{\text{tpl}}$ by letting $V_w(c) = \mathcal{S}_w(c)$. The empty $\mathcal{S}_w$-valuation extended to
Terms will also be denoted by \( V^\emptyset \) (and will likewise be referred to as the empty \( S_w \)-valuation) though the extended domain includes (only) the six names in the language. Then, we have the following standard definition of what it means for an (extended) \( S_w \)-valuation \( V_w \) to satisfy a formula \( \varphi \) in a given structure \( S_w \):

- \( S_w, V_w \models P^1(t) \iff V_w(t) \in S_w(P^1) \)
- \( S_w, V_w \models P^2(t_1, t_2) \iff \langle V_w(t_1), V_w(t_2) \rangle \in S_w(P^2) \)
- \( S_w, V_w \models P^3(t_1, t_2, t_3) \iff \langle V_w(t_1), V_w(t_2), V_w(t_3) \rangle \in S_w(P^3) \)
- \( S_w, V_w \not\models \perp \)
- \( S_w, V_w \models \varphi \to \psi \iff S_w, V_w \models \varphi \) only if \( S_w, V_w \models \psi \)
- \( S_w, V_w \models \forall x \varphi \iff S_w, V_w^{x/u} \models \varphi \) for all \( u \in U_w \)

where \( V_w^{x/u} = V_w \) except that \( V_w^{x/u}(x) = u \). Then, a sentence \( \varphi \) in \( L_{lpl} \) is true in the structure \( S_w \) just in case it is satisfied by the empty \( S_w \)-valuation \( V_w^\emptyset \). This is all elementary, but the notation is presented here to set the stage for what follows.

2.3. The sorted monadic first-order language \( L_{0\square}^* \)

We begin with much the same language as above, \( L_{lpl} \), except that all predicate symbols will be unary. Each predicate symbol \( P^s \) in \( L_{lpl} \) appears in \( L_{0\square}^* \) except that the infix binary identity symbol ‘=’ in \( L_{lpl} \) will be replaced with a prefix unary predicate symbol ‘Equal’ in \( L_{0\square}^* \). The predicate symbols will continue to be distinguished (sorted) by their arity in \( L_{lpl} \) (marked in some cases by a superscript), but in \( L_{0\square}^* \) they are all unary predicate symbols. The grammar will be revised accordingly (yielding only a fragment of what is possible more generally):

\[
\begin{align*}
t^1 & ::= \langle c \rangle \mid \langle x \rangle \\
t^s & ::= \langle t_1, \ldots, t_i \rangle \quad \text{for } s \in \mathbb{Z}^+ \\
\varphi & ::= P^1(t^1) \mid P^2(t^2) \mid P^3(t^3) \mid \perp \mid \varphi_1 \to \varphi_2 \mid \forall x \varphi
\end{align*}
\]

Here \( \langle t_1, \ldots, t_i \rangle \) denotes the single entity consisting of the ordered \( i \)-tuple where, by definition,

\[
\begin{align*}
\langle t_1 \rangle & \overset{\text{def}}{=} t_1 \\
\langle t_1, t_2 \rangle & \overset{\text{def}}{=} \{ \{ t_1 \}, \{ t_1, t_2 \} \} \\
\langle t_1, \ldots, t_{i+1} \rangle & \overset{\text{def}}{=} \langle \langle t_1, \ldots, t_i \rangle, t_{i+1} \rangle
\end{align*}
\]
For instance, \( \langle t_1, t_1 \rangle \) is \( \{ \{ t_1 \} \} \), in contrast with either \( \{ t_1 \} \) or \( \langle t_1 \rangle \). This notation provides a way to map formulas in \( L_{lpl} \) one-to-one to formulas in \( L^* \), where a predicate symbol \( P^s \) in \( L_{lpl} \) will have \( s \) arguments while the corresponding predicate symbol in \( L^* \) will have just one argument.

There are infinitely-many sorts of terms \( t^s \) in \( L^* \)—denoted as \( \text{Terms}_0 \) for \( s \in \mathbb{Z}^+ \). Nevertheless \( L^* \) uses just one of the infinitely-many respective quantifier symbols \( \forall^s \); namely, \( \forall \) is \( \forall^1 \).

Any given structure \( S_w \) for \( L_{lpl} \) can be straightforwardly modified to build a structure \( S^*_w \) for \( L^* \), as follows. Let \( U^*_w = U_w \cup U^*_w \cup U^*_w \cup \ldots \) be the collection of all ordered tuples of blocks in the world \( w \). The universe \( U^*_w \) is thus constituted by infinitely-many disjoint sorts, \( U^1_w, U^2_w, U^3_w, \) etc., and elements from only the first three of these sorts \( U^s_w \) are appropriate for respective predicate symbols \( P^s \). Any structure \( S^*_w \) for \( L^* \) must then be such that

- \( S^*_w(\forall) = U^1_w \subseteq U^*_w \)
- \( S^*_w(c) \subseteq U^1_w \subseteq U^*_w \) for each name \( c \)
- \( S^*_w(P^1) \subseteq U^1_w \subseteq U^*_w \) for each of six sort-1 predicate symbols \( P^1 \)
- \( S^*_w(P^2) \subseteq U^2_w \subseteq U^*_w \) for each of twelve sort-2 predicate symbols \( P^2 \)
- \( S^*_w(P^3) \subseteq U^3_w \subseteq U^*_w \) for the one sort-3 predicate symbol \( P^3 \)

The extensions of predicate symbols in \( L^* \) are sets of ordered \( s \)-tuples where each such \( s \)-tuple is regarded as a single individual.

Traditionally, the extensions of corresponding predicate symbols in \( L_{lpl} \) are also denoted as sets of ordered \( i \)-tuples as a way of distinguishing \( s \) individuals taken in a specific order as arguments. The notation for specifying structures for these two languages does not capture a crucial distinction. The distinction should be acknowledged nonetheless. Though the extensions of respective predicate symbols in the two languages are listed as sets of ordered tuples, the intended meanings of the predicate symbols in \( L_{lpl} \) apply directly to the \( \epsilon \)-ancestral atoms of those tuples (taken in a specific order) whereas the intended meanings of the predicate symbols in \( L^* \) apply directly to the ordered tuple itself, regarded perhaps as an ordered mereological sum, if that makes sense. This distinction, while not reflected in the notation above for specifying structures \( S_w \) and \( S^*_w \), is reflected in the grammar where \( P^s \) in \( L^* \) takes the ordered \( s \)-tuple itself as its one argument while the corresponding \( P^s \) in \( L_{lpl} \) takes (in the specified order) the \( \epsilon \)-ancestral atoms in the ordered \( s \)-tuple as its \( s \) arguments. In a sense, structures for \( L^* \) are truer to what it means to say, literally, that the extension of a binary predicate symbol
(say) is a set of ordered pairs. In that case, any first-order language may be regarded as being monadic in this literal sense.

Quantification only over sort-1 entities is allowed in this language fragment, though we also allow quantification into sort-$s$ terms for $s > 1$. Specifically, let $\text{Variables}_{L_{\square\square}}^1$ be the set of sort-1 variables $x, y, \ldots$. Then, an $S_w^*$-valuation will be a total or partial (possibly empty) function

$$V_w^*: \text{Variables}_{L_{\square\square}}^1 \to U_w^1$$

assigning a block $V_w^*(x)$ in $U_w^1$ to each sort-1 $L_{\square\square}$-variable $x$ in its domain. Let $V_w^*$ denote the empty $S_w^*$-valuation. The function $V_w^*$ extends uniquely to $\text{Terms}_{L_{\square\square}}^1$ (the set of sort-1 terms) by letting $V_w^*(c) = S_w^*(c)$. The empty $S_w^*$-valuation extended to $\text{Terms}_{L_{\square\square}}^1$ will again be denoted by $V_w^*$ (and will likewise be referred to as the empty $S_w^*$-valuation) though the extended domain includes (only) the six names in the language. The function $V_w^*$ also extends uniquely to $\text{Terms}_{L_{\square\square}}^s$ (for $s > 1$) by letting $V_w^*((t_1^1, \ldots, t_s^1)) = (V_w^*(t_1^1), \ldots, V_w^*(t_s^1))$. Then, we have the following standard definition of what it means for an (extended) $S_w^*$-valuation $V_w^*$ to satisfy a formula $\varphi$ in a given structure $S_w^*$:

- $S_w^*, V_w^* \models P^1(t^1)$ iff $V_w^*(t^1) \in S_w^*(P^1)$
- $S_w^*, V_w^* \models P^2(t^2)$ iff $V_w^*(t^2) \in S_w^*(P^2)$
- $S_w^*, V_w^* \models P^3(t^3)$ iff $V_w^*(t^3) \in S_w^*(P^3)$
- $S_w^*, V_w^* \not\models \bot$
- $S_w^*, V_w^* \models \varphi \rightarrow \psi$ iff $S_w^*, V_w^* \models \varphi$ only if $S_w^*, V_w^* \models \psi$
- $S_w^*, V_w^* \models \forall x \varphi$ iff $S_w^*, V_w^*\times/u \models \varphi$ for all $u \in U_w^1$

where, again, $V_w^*\times/u = V_w^*$ except that $V_w^*\times/u(x) = u$. A sentence $\varphi$ in $L_{\square\square}^*$ is true in the structure $S_w^*$ just in case it is satisfied by the empty $S_w^*$-valuation $V_w^*\emptyset$.

**Claim 1:** The language $L_{lpl}$ is semantically isomorphic to $L_{\square\square}^*$.

This trivial fact perhaps calls for some notational clarity. First, for a first-order language $L$,

- $\text{Structures}_L$ denotes the set of structures for a first-order language $L$.
- $\text{Valuations}_L$ denotes the set of valuations for $L$.
- $\text{Formulas}_L$ denotes the set of formulas in $L$. 

10
• \( \text{Terms}_L \) denotes the set of terms in \( L \).
• \( \text{Predicates}_L \) denotes the set of predicate symbols in \( L \).
• Etc.

Then two first-order languages \( L \) and \( L' \) will be said to be \textit{semantically isomorphic} just in case there exist bijective mappings

• \( \phi : \text{Formulas}_L \rightarrow \text{Formulas}_{L'} \)
• \( \mu_\phi : \text{Structures}_L \times \text{Valuations}_L \rightarrow \text{Structures}_{L'} \times \text{Valuations}_{L'} \)

such that

\[ S_L, V \models \varphi \text{ iff } \mu_\phi(S_L, V) \models \phi(\varphi). \]

Then, to prove Claim 1, let

\[ \iota : \text{Terms}_{L_{ppl}} \cup \text{Predicates}_{L_{ppl}} \rightarrow \text{Terms}_{L_{0\Box}} \cup \text{Predicates}_{L_{0\Box}} \]

be a mapping where

• \( \iota(t) = t \)
• \( \iota(P^*) = P^* \) except that \( \iota(=) = \text{Equal} \)

This is essentially an identity relation on the terms and predicate symbols of \( L_{ppl} \) given that \( \text{Terms}_{L_{ppl}}^{1\Box} = \text{Terms}_{L_{ppl}} \) and \( \text{Predicates}_{L_{ppl}}^{1\Box} = \text{Predicates}_{L_{ppl}} \) except for the respective identity predicate symbols.

Define \( \phi : \text{Formulas}_{L_{ppl}} \rightarrow \text{Formulas}_{L_{0\Box}} \) such that

• \( \phi(P^1(t)) = \iota(P^1(\iota(t))) = P^1(t) = P^1(\langle t \rangle) \)
• \( \phi(P^2(t_1, t_2)) = \iota(P^2(\iota(t_1), \iota(t_2))) = \iota(P^2(\langle t_1, t_2 \rangle)) \)
• \( \phi(P^3(t_1, t_2, t_3)) = \iota(P^3(\iota(t_1), \iota(t_2), \iota(t_3))) = P^3(\langle t_1, t_2, t_3 \rangle) \)
• \( \phi(\bot) = \bot \)
• \( \phi(\varphi_1 \rightarrow \varphi_2) = \phi(\varphi_1) \rightarrow \phi(\varphi_2) \)
• \( \phi(\forall x \varphi) = \forall \iota(x) \phi(\varphi) = \forall x \phi(\varphi) \)

Also define

\[ \mu_\phi : \text{Structures}_{L_{ppl}} \times \text{Valuations}_{L_{ppl}} \rightarrow \text{Structures}_{L_{0\Box}} \times \text{Valuations}_{L_{0\Box}} \]
where \( \mu_\varphi(S_w, V_w) = \langle S_w^*, V_w^* \rangle \) as specified earlier. A standard inductive proof shows that \( S_w, V_w \models \varphi \) iff \( \mu_\varphi(S_w, V_w) \models \varphi(\varphi) \) for any \( L_{tpl} \)-formula \( \varphi \). This is the sense in which \( L_{tpl} \) is semantically isomorphic to \( L_{\square}^* \).

Note, in particular, that \( \varphi(\varphi) \).

Claim 2: The language \( L_{\square}^* \) is a first-order correspondence language for the following “sorted” modal sentential language, \( L_{\square}^* \).

2.4. The modal sorted sentential language \( L_{\square}^* \)

To quickly review: given a collection of sentence symbols \( \{p, q, \ldots\} \), a single modal symbol \( \square \), and the small but adequate set of sentential connectives \( \{\bot, \to\} \), we can define a simple modal sentential language \( L \) as follows:

\[
\varphi ::= p \mid \bot \mid \varphi_1 \to \varphi_2 \mid \square \varphi
\]

A Kripke model \( \mathcal{M} = \langle W, R, V \rangle \) for \( L \) provides a means to interpret a particular kind of classical first-order language. Namely, \( W \) may serve as a domain of quantification while \( R \) specifies the extension in \( W \times W \) of a single respective binary relation symbol \( R \). Likewise, to each \( L \)-sentence symbol \( p \) there may be associated a unary predicate symbol \( P \) whose extension is \( V(p) \). The resulting language \( L^* \) is a so-called first-order correspondence language for \( L \). Again, a standard inductive translation

\[
*_{x} : \text{Formulas}_{L} \times \text{Variables}_{L^*} \to \text{Formulas}_{L^*}
\]

goes as follows:

\[
\begin{align*}
p^*_x &= P(x), \\
\bot^*_x &= \bot, \\
(\varphi \to \psi)^*_x &= \varphi^*_x \to \psi^*_x, \\
(\square \varphi)^*_x &= \forall x (R(x, y) \to \varphi^*_y)
\end{align*}
\]

Every \( L \)-formula \( \varphi \) corresponds to a set of first-order \( L^* \)-formulas \( \varphi^*_x \) by way of this translation; and for any model \( \mathcal{M} \) for \( L \), there is a structure \( S_{2\mathcal{M}} \) for \( L^* \) such that

\[
\mathcal{M}, u \models \varphi \quad \text{iff} \quad S_{2\mathcal{M}} \models \varphi^*_x[x \leftarrow u]
\]

The structure \( S_{2\mathcal{M}} \) is in fact just \( \mathcal{M} \), or a transcription of \( \mathcal{M} \), where \( W_{2\mathcal{M}} \) is the domain of quantification, \( R_{2\mathcal{M}} \) specifies the extension of \( R \) in \( W_{2\mathcal{M}}^2 \), and \( V(p) \) is the extension of \( p^* \) in \( V_{2\mathcal{M}} \). This correspondence provides a connection between modal sentential logic and classical first-order logic such that meta-theoretic results for the latter transfer to the former [30, 32].
Note, again, that correspondence theory in such a vein starts with a given
modal sentential language and constructs the respective first-order correspon-
dence language accordingly. In the present case, we need to work in the other
direction. We already have a sorted monadic first-order language $L_0$ and
want to find a modal sentential language $L_0$ such that similar constructions
would yield such a first-order language as a respective first-order correpon-
dence language. Since $L_0$ is a many-sorted (and thus non-standard) lan-
guage, we should not be surprised if we end up with a non-standard language
$L_0$ that is itself sorted in some sense.

Indeed, given that we have a first-order language with sorted predicate
symbols, we should expect its modal sentential reverse-correspondence lan-
guage to include a collection of what may reasonably be called sorted atomic
formula-types $\{p^1, q^1, \ldots\} \cup \{p^2, q^2, \ldots\} \cup \{p^3, q^3, \ldots\}$. Atomic formula-types
are not yet full-fledge formulas for reasons given below, but a formula will
have the form $p^{s}_i$ where the $|a|$ is a needed reference-fixing device. With
these ingredients, in addition to a modal symbol $\Box$ and a small but ade-
quate set of sentential connectives $\{\bot, \to\}$, we will define the sorted modal
sentential language $L_0$ as follows:

$$\varphi ::= p^1_i \mid p^2_{i,j} \mid p^3_{i,j,k} \mid \bot \mid \varphi_1 \to \varphi_2 \mid \Box \varphi$$

The formulas will be sorted (indicated by a superscript) in the sense that
each may have a truth-value only in a given sort of “world.”

The reference-fixing device is needed given that an atomic formula of sort
$s$ often will need to have a truth-value in a model at a world $\langle u_1, \ldots, u_n \rangle$
for $n \neq s$. Formulas in this case will have a truth-value in such a world
only by indicating the part of the world to which it applies (especially if
$n > s$)—a part that is in its own right a world. That is, worlds will be
ordered composites of certain basic (sort-1) worlds. In this regard, all atomic
formulas in $L_0$ are “essentially indexical” [48] in the sense that each use of
the formula in a given world, to have content, requires some indication of the
part of the world to which it applies.

For instance, the notation $|a|$ when used with reference to a world
$\langle u_1, \ldots, u_n \rangle$ with $1 \leq i, j \leq n$ is thus shorthand for a projection function
that picks out only the sort-1 worlds in the $i^{th}$ and $j^{th}$ positions in $\langle u_1, \ldots, u_n \rangle$
to supply (in that order) the concrete reference (in this case, binary) of the
atomic formula-type to which it is appended. The references of formulas in
this case are not truth-values but rather the worlds in which (with reference
to which) they have whatever truth-value they have there.

Why all this bother? Perhaps we should back up a bit. Let $w$ be a fixed
target-level blocks world, and let $U_w$ be the collection of all of the blocks in
$w$ with all of their properties and relations. Ordered $n$-tuples of these
blocks \((1 \leq n \leq 3)\) are the objects that appear in \(L^*_{\Box}\)-models as possible referents of the monadic terms in that language. These objects correspond to what will serve as source-level \textit{worlds} in models for \(L_{\Box}\). In either case, we will want to be able to formulate single formulas referring to an arbitrary number of target-level blocks or source-level worlds (e.g., to state that if one block is between two other blocks, then a fourth block is between yet two other blocks) or single formulas that involve more than one sort of object (e.g., to state that if one block is a cube and so is another, then the two have the same shape). The two constituent formulas in the first conditional use a common predicate symbol but with reference to different triples, while the second statement refers to two blocks in a couple of different ways, either separately or as a single pair. Thus, if we want to be able to state in general, for any \(L_{\Box}\)-model \(M\), that the respective \(L^*_{\Box}\)-structure \(S^*_{M}\) is such that
\[
M, u \models \varphi \iff S^*_{M} \models \varphi^*_{x \leftarrow u},
\]
then any number of examples present difficulties. For instance, \(L^*_{\Box}\) has nineteen unary predicate symbols of various sorts, so let \(P^1_2\) and \(P^2_7\) be the second and seventh of these predicate symbols where the second is of sort-1 and the tenth is of sort-2. Let \(\psi_{x,y}\) be the formula
\[
( P^1_2(\langle x \rangle) \land P^1_2(\langle y \rangle) ) \rightarrow P^2_7(\langle x, y \rangle)
\]
For instance, let ‘\(P^1_2\)’ be \textit{Cube}, and let ‘\(P^2_7\)’ be \textit{Sameshape}. Then, assume there is a structure \(S^*\) for \(L^*_{\Box}\) such that
\[
S^* \models \psi_{x,y} [x \leftarrow u_1] [y \leftarrow u_2]
\]
This says that the given formula \(\psi_{x,y}\) is satisfiable by \(\langle u_1, u_2 \rangle\) relative to structure \(S^*\).

So here is the issue. Is there an \(L_{\Box}\)-formula \(\varphi\) and an \(L_{\Box}\)-model \(M\) such that its standard translation \(\varphi^*\) is \(\psi_{x,y}\) and \(S^*\) is \(S^*_{M}\)? Apparently not, or at least not as written. Namely, the sorted aspects of the language \(L^*_{\Box}\) present difficulties for the standard translation method for formulas in a basic modal language. Yet because of the particular nature of the sorting (by arities), the standard translation method can be modified to accommodate \(L^*_{\Box}\)-formulas like \(\psi_{x,y}\).

The strategy will be as follows. First, we note the following facts and stipulations.

- **Predicate symbols and formula-types:** Since there are nineteen predicate symbols \(P^s_l\) of three different sorts (arities) in \(L^*_{\Box}\) (that is, with \(1 \leq s \leq 3\) and \(1 \leq l \leq 19\)), we need nineteen atomic formula-types \(p^s_l\) in \(L_{\Box}\), respectively. The subscript \(l\) numbers the nineteen predicate symbols in \(L^*_{\Box}\) and respective atomic formula-types in \(L_{\Box}\).
• **Sorts:**

- \(L_0^*\)-**predicate symbols:** Each predicate symbol \(P^*_i\) in \(L_0^*\) will be of some fixed sort \(s\) (for \(1 \leq s \leq 3\)) which is identified with the arity of its \(s\)-tuple argument.

- \(L_0^*\)-**formula-types and formulas:** The respective atomic formula-type \(p^*_s\) in \(L_0^*\) is also said to be of sort \(s\) (again, with \(1 \leq s \leq 3\)), indicating the unique sort of *world* with reference to which formulas of that type can have a truth-value.

- \(S^*_w\)-**objects:** In any given structure \(S^*_w\) for \(L_0^*\), the domain of objects will be the union of \(s\)-tuples

\[
U_w^* = \bigcup \{U_w^s \mid s \in \mathbb{Z}^+\} = U_w^1 \cup U_w^2 \cup U_w^3 \cup U_w^4 \cup \ldots ,
\]

where \(U_w^1\) is the set of (at least one and at most twelve) blocks in a given blocks world \(w\). Any given object \(\langle u_1, \ldots, u_s \rangle \) in \(U_w^*\) is of sort \(s\) (for \(s \in \mathbb{Z}^+\)).

- \(M_w\)-**worlds:** In any given model \(M_w = \langle W_w, R_w, V_w \rangle\) for \(L_0^*\), the domain \(W_w\) of worlds will be the same union of \(n\)-tuples

\[
U_w^* = \bigcup \{U_w^n \mid n \in \mathbb{Z}^+\} = U_w^1 \cup U_w^2 \cup U_w^3 \cup U_w^4 \cup \ldots ,
\]

where \(U_w^1\) is the set of (up to twelve) sort-1 worlds determined by \(w\). This set of source-worlds is, again, just the set of ordered tuples of *blocks* (under a different description) in target-world \(w\). Any given \(M_w\)-world \(\langle u_1, \ldots, u_n \rangle \) in \(U_w^*\) is of sort \(n\) (for \(n \in \mathbb{Z}^+\)).

• **\(L_0^*\)-formulas:** An atomic \(L_0^*\)-formula is of the form \(p^*_s \| \vec{a}\) where \(\vec{a}\) designates the part of the given world to which the formula-type \(p^*_s\) refers (thus yielding an \(\vec{a}\)-token of the formula-type). So there will be many more than nineteen atomic formulas in \(L_0^*\). Formulas in \(L_0^*\) will be evaluated in worlds \(\langle u_1, \ldots, u_n \rangle \in W_w\) where the world may be of a higher sort than that of the given formula; thus the formula itself (essentially indexical in this sense), to be truth-evaluable, must include some indication of the proper part of the world to which it refers.

• **Translation terms:** \(\vec{t}\) denotes an ordered \(n\)-tuple of sort-1 terms in \(Terms_{L_0^*}^1\), the latter including both names \(c\) and variables \(x\).

• **Projection indices:** The reference-fixing mechanism \(\| \vec{a}\) when appended to a formula-type and used in a world \(\langle u_1, \ldots, u_n \rangle \) is essentially a projection function \(\pi_{\vec{a}} : U_w^* \rightarrow U_w^{\| \vec{a}\}}\) that picks out an \(\| \vec{a}\)-tuple of sort-1 worlds in \(\langle u_1, \ldots, u_n \rangle \) in positions designated by \(\vec{a}\) (as a 1-, 2-, or 3-tuple), indicating (in that order) the concrete reference of the formula-type to which it is appended. Namely,
$\vec{a} = \begin{cases} 
\langle i \rangle & \text{for positive } i \leq n \quad \text{if } s = 1 \\
\langle i, j \rangle & \text{for positive } i, j \leq n \quad \text{if } s = 2 \\
\langle i, j, k \rangle & \text{for positive } i, j, k \leq n \quad \text{if } s = 3 
\end{cases}$

and $\pi_{\vec{a}}(\langle u_1, \ldots, u_n \rangle) \in U^{|\vec{a}|}_w$ where $|\vec{a}|$ denotes the length of $\vec{a}$ ($1 \leq |\vec{a}| \leq 3$). There will be less of a mess here if the projection tuples $\vec{a}$ are written without the angle brackets. (Examples are presented below.)

Given these facts and stipulations, recall that $L_{0\Box}$ is defined as follows:

$$\varphi ::= p^1_i \mid p^2_{i,j} \mid p^3_{i,j,k} \mid \bot \mid \varphi_1 \rightarrow \varphi_2 \mid \Box \varphi$$

Then, any S5 model $M_w = \langle W_w, R_w, V_w \rangle$ for $L_{0\Box}$ will be such that

- $W_w$ is $U^*_w$, the set of all ordered $n$-tuples of sort-1 worlds in $U^1_w$;
- $R_w$ is the universal binary relation on $W_w$ (namely, for any $u, v \in W_w, \langle u, v \rangle \in R_w$); and
- $V_w : AtomicFormulaTypes_{L_{0\Box}} \rightarrow 2^{W_w}$ is defined such that $V_w(p^s) \subseteq U^*_w \subseteq V(p)$ (which could be regarded as the “character” of the formula-type $p^s$ relative to $M_w$).

We inductively define the notion of an $L_{0\Box}$-formula being $M_w$-true at a world $\vec{u} = \langle u_1, \ldots, u_n \rangle$, for any $n \in \mathbb{Z}^+$, as follows:

- $M_w, \vec{u} \models p^s|_{\vec{a}}$ if $\pi_{\vec{a}}(\vec{u}) \in V_w(p^s)$
- $M_w, \vec{u} \not\models \bot$
- $M_w, \vec{u} \models \varphi \rightarrow \psi$ if $M_w, \vec{u} \models \varphi$ only if $M_w, \vec{u} \models \psi$
- $M_w, \vec{u} \models \Box \varphi$ if $M_w, \vec{u} \models \varphi$ for all $\vec{v} \in U^*_w$

Let

$$X_w \overset{\text{def}}{=} \{ \langle \vec{u}, p^s \rangle \mid \vec{u} \in V_w(p^s) \}$$

This is a unique relation in $W_w \times AtomicFormulaTypes_{L_{0\Box}}$ determined by $V_w$. The set $X_w(\vec{u}) = \{ p^s \mid \vec{u} \in V_w(p^s) \}$ is the set of all $L_{0\Box}$-formula-types $p^s$ such that $p^s|_{\vec{a}}$ is $M_w$-true in $\vec{u}$, where

- $\vec{a}$ is $\langle 1 \rangle$ if $s = 1 = |\vec{u}|$,
- $\vec{a}$ is $\langle 1, 2 \rangle$ if $s = 2 = |\vec{u}|$, and
- $\vec{a}$ is $\langle 1, 2, 3 \rangle$ if $s = 3 = |\vec{u}|$. 

16
The set $X_w(u)$ is what we may refer to as the “characterization” of $u$ relative to $M_w$. Note that the domain of $X_w$ (which is the range of $V_w$) is just $U^1_w \cup U^2_w \cup U^3_w$, not the whole of $\mathcal{W}_w$.

We are now in a position to define a correspondence translation

$$*_{\bar{t}} : L_{0\square} \rightarrow L^*_{0\square},$$

that is standard in some but not all respects. For $\bar{t}$ an ordered $n$-tuple of sort-1 terms in $\text{Terms}_{L_{0\square}}^1$ (including both names $c$ and variables $x$), let $\bar{t}' = (t'_1, \ldots, t'_n)$ be such that any component $t'_i$ is a variable different from any component $t_j$ of $\bar{t}$ if $t_i$ is a variable, and otherwise is identical with $t_j$ if the latter is a name. Then, for $1 \leq l \leq 19$, for each $L_{0\square}$-formula-type $p^*_l$ there is a corresponding $L^*_0$-predicate-symbol $P^*_l$ such that

$$(p_l^* | a)^x_{\bar{t}} = P_l^*(\pi_\bar{a}(\bar{t})),
\downarrow l^*_x = \downarrow l,
(\varphi \rightarrow \psi)^x_{\bar{t}} = \varphi^x_{\bar{t}} \rightarrow \psi^x_{\bar{t}},
(\Box \varphi)^x_{\bar{t}} = \forall \bar{t}'(\varphi^x_{\bar{t}})
$$

where $\bar{t}'$ is as specified above, and where ‘$\forall \bar{t}$’ is shorthand for ‘$\forall t_1, \ldots, \forall t_n$’.

What we want to be able to show then is that

$$\mathcal{M}_w, \bar{u} \models \varphi \iff S^*_w \models \varphi^*_x(\bar{t}) [\bar{t} \leftarrow \bar{u}]$$

A straightforward inductive proof establishes this result if it is the case (or if we stipulate) that $V_w(p^*_l)$ specifies the extension of $P^*_l$ for each of the nineteen $L_{0\square}$-formula-types $p^*_l$ and respective unary $L^*_0$-predicate symbols $P^*_l$—which, of course, is consistent with the usual construction of the first-order correspondence language for a standard modal sentential language.

Normally, so far as correspondence theory goes, one is given the modal sentential language and then defines the first-order correspondence language accordingly so that such a proof is possible. Here things are reversed. We are given a first-order language $L^*_0$ with interpreted predicate symbols. Namely, there are nineteen predicate symbols in $L_{1pl}$ and thus in $L^*_0$. The correspondence of the first-order correspondence language $L^*_0$ to a reverse-engineered modal sentential language $L_{0\square}$ would thus (one would think) depend on the fact that the material contents of atomic $L_{0\square}$-formulas should reflect the material contents of respective predicate symbols in $L^*_0$.

For instance, if it is the case that $S^*_w \models \text{Cube}(c)$ (where ‘$\text{Cube}$’ is $P^{1}_2$), then the corresponding formula-type $p^*_1$ should itself have a meaning reflecting the fact that the sort-1 world $u$ such that $S^*_w(c) = u$ is “cube-like” in nature.
Granted, one might develop a modal sentential language $L_{0\Box}$ where it is possible for the atomic formulas themselves to refer to the worlds in which they are true or not. In that case, $p^1_2$ might simply say that “this world is cube-like”—immediately reflecting what ‘Cube(c)’ says about the block named ‘c’. This is a trivial way to interpret the formulas in $L_{0\Box}$ that yields nothing more informative than what we already have (if anything) in the contents of $L_{0\Box}^*$ predicate symbols. We will have instituted formal changes with no material effects. Does that matter?

At this point we should recall ideas (2) and (4) (page 2). In speculating that classes of models for ability-based dynamic languages can be used in place of classes of things to interpret the predicate symbols in a first-order language, one of the goals is to show that the notion of material inference may be cashed out in terms of relationships among such classes of models. Such models should in some way say what it is for a source-level world to be cube-like rather than simply state that that is so with no further explanation. In order to focus, then, on the details in such worlds, it would help to think of them as worlds consisting of source-level objects (aspects of blocks rather than the blocks themselves) having various properties and standing in various relations. In this sense, a given cube as a target-level object is a source-level world consisting of various source-level objects like faces, edges, vertices, etc., standing in various source-level relations to one another. The latter provide the ingredients for so-called “characterizations” of worlds that, in a different light, are just blocks.

This suggests that the distinction between target and source levels ought not be merely formal. The source level should supply information in a kind of detail that is not available at the target level as such. This opens up the possibility of a semantic/informational hierarchy of levels of detail extending in both directions. A change of levels in such a hierarchy should be material, whereas merely formal changes effect only lateral moves in the hierarchy.

If it is the case that $S^*_u \models \text{Cube}(c)$ where ‘Cube’ is $P^1_2$, then the corresponding formula-type $p^1_2$ should itself have a more informative if not explanatory meaning that does not merely state that the sort-1 world $u$ such that $S^*_u(c) = u$ is “cube-like” in nature but rather says what such a nature amounts to.

This exposes a wrinkle that needs to be ironed out. There are many candidate atomic formulas-types $p^1_2$ available to provide informative “characterizations” of cube-like worlds $u \in W_u$. There are far more than nineteen candidate atomic formula-types $p^1_2$ available to provide informative characterizations of the nineteen different kinds of worlds in $W_u$. Yet, we want to specify only nineteen atomic formula-types (one per predicate symbol in $L_{0\Box}^*$) such that respective formulas may be true or false in these worlds.
There are several ways to do this.
For instance, we might require only *extensional sufficiency*.

So far as sort-1 worlds are concerned, we may treat any one distinctive aspect of blocks—their faces, say—as objects in such worlds \( u \); and we may distinguish three possible properties of such objects, e.g., their numbers of edges. Without getting into too many details, we will then be able to formulate three distinct atomic formula-types \( p_1^1, p_2^1, p_3^1 \) that correspond *materially* to the three respective shape predicates in \( L^*_0 \). Formulas involving three such atomic formula-types would be:

- \( p_1^1_i \): “All faces here \( i \) have three edges.”
- \( p_2^1_i \): “All faces here \( i \) have four edges.”
- \( p_3^1_i \): “All faces here \( i \) have five edges.”

Notice that the particular atomic formula-types \( p_i^s \), as with many alternative atomic formula-types, are easily *operationalized* by way of edge-counting operations. That is where the present discussion will connect with operationalist pragmatism [28]. In any case, we have

- \( (p_1^1_i)_{\vec{t}} = P_1^1(\pi_i(\vec{t})) = Tet(\pi_i(\vec{t})): \) “The object denoted by the \( i^{th} \) component of \( \vec{t} \) is a tetrahedron.”
- \( (p_2^1_i)_{\vec{t}} = P_2^1(\pi_i(\vec{t})) = Cube(\pi_i(\vec{t})): \) “The object denoted by the \( i^{th} \) component of \( \vec{t} \) is a cube.”
- \( (p_3^1_i)_{\vec{t}} = P_3^1(\pi_i(\vec{t})) = Dodec(\pi_i(\vec{t})): \) Etc.

More generally, objects in sort-1 worlds could include various facets of individual blocks \( u \) in \( w \): *faces, hulls, tops, grid-locations*, if not edges, vertices, shapes, heights, widths, and so forth. *Faces* will have various properties—numbers of vertices, numbers of edges, particular shapes, and so on. The properties of *hulls* (closed three-dimensional surfaces enclosing convex and thus connected volumes) will include numbers of faces, numbers of vertices, widths, heights, and so on. The *tops* of sort-1 worlds (one per world) will have various shapes and heights above the grid. The *grid-location* in a sort-1 world (also one per world) will have various properties including its row and its column (see Appendix B). Etc. All such properties are easily operationalized in terms of measurement procedures, so to speak. The idea, then, is that (atomic) facts about such objects in a given sort-1 world should constitute *features* of respective blocks in \( w \). The state-description of \( u \) as a *world* constitutes the characteristics of \( u \) as a *tuple of blocks* in \( w \).
We also need to select three distinct atomic formula-types for \( L_{0\Box} \) that correspond uniquely to the three respective size predicates in \( L_{tpl} \). Respective formulas might be, for instance:

- \( p^1_i \): “The height of the single top here \( i \) is less than a ‘squarewidth’.”
- \( p^5_i \): “The height of the single top here \( i \) is equal to a ‘squarewidth’.”
- \( p^6_i \): “The height of the single top here \( i \) is greater than a ‘squarewidth’.”

A “squarewidth” is a fixed unit of length equal to the width of a square on the grid surface. Determining such heights is easily conceived in operational terms. In this case, we have

- \( (p^1_i)_{\vec{t}} = P^1_{\pi_i(\vec{t})} = Small(\pi_i(\vec{t})) \)
- \( (p^5_i)_{\vec{t}} = P^1_{\pi_i(\vec{t})} = Medium(\pi_i(\vec{t})) \)
- \( (p^6_i)_{\vec{t}} = P^1_{\pi_i(\vec{t})} = Large(\pi_i(\vec{t})) \)

This takes care of all six unary predicate symbols in \( L_{tpl} \) and respective sort-1 unary predicate symbols in \( L^*_{0\Box} \).

To do likewise for the twelve binary predicate symbols in \( L_{tpl} \) and respective sort-2 unary predicate symbols in \( L^*_{0\Box} \), we need twelve more atomic formulas-types describing distinctive features of the objects in sort-2 worlds \( u \) in \( U^2_{w} \). As ordered pairs of blocks in \( w \), such worlds will include the objects in sort-1 worlds (faces, tops, grid-locations, etc.) if not other “relational” objects (besides grid-location, e.g., distances between blocks, differences in heights, etc.). To be brief, the following set of twelve respective atomic formulas should suffice:

- \( p^7_{i,j} \): “All faces here \( i,j \) have the same number of edges.”
- \( p^8_{i,j} \): “The first \( i \) top is higher than the second \( j \) top.”
- \( p^9_{i,j} \): “The first top \( i \) is lower than the second \( j \) top.”
- \( p^{10}_{i,j} \): “The two \( i,j \) tops have the same height.”
- \( p^{11}_{i,j} \): “The column-number of the first \( i \) grid-location is less than the column-number of the second \( j \) grid-location.”
- \( p^{12}_{i,j} \): “The column-number of the first \( i \) grid-location is greater than the column-number of the second \( j \) grid-location.”
• \( p_{13}^2(i,j) \): “The column-number of the first grid-location is the same as the column-number of the second grid-location.”

• \( p_{14}^2(i,j) \): “The row-number of the first grid-location is less than the row-number of the second grid-location.”

• \( p_{15}^2(i,j) \): “The row-number of the first grid-location is greater than the row-number of the second grid-location.”

• \( p_{16}^2(i,j) \): “The row-number of the first grid-location is the same as the row-number of the second grid-location.”

• \( p_{17}^2(i,j) \): “The distance between the two grid-locations is exactly one squarewidth.”

• \( p_{18}^2(i,j) \): “The first grid-location is the same as the second grid-location.”

The grid-location of a “world” will be specified numerically so that the column- and row-numbers of a given grid-location may be read from Table B.4 (page 37). In this case, we have

\[
\begin{align*}
(p_{13}^2(i,j))^* & = P_{13}^2(\pi_{i,j}(\vec{t})) = \text{SameShape}(\pi_{i,j}(\vec{t})) \\
(p_{14}^2(i,j))^* & = P_{14}^2(\pi_{i,j}(\vec{t})) = \text{SameSize}(\pi_{i,j}(\vec{t})) \\
(p_{15}^2(i,j))^* & = P_{15}^2(\pi_{i,j}(\vec{t})) = \text{BackOf}(\pi_{i,j}(\vec{t})) \\
(p_{16}^2(i,j))^* & = P_{16}^2(\pi_{i,j}(\vec{t})) = \text{SameRow}(\pi_{i,j}(\vec{t})) \\
(p_{17}^2(i,j))^* & = P_{17}^2(\pi_{i,j}(\vec{t})) = \text{Adjoins}(\pi_{i,j}(\vec{t})) \\
(p_{18}^2(i,j))^* & = P_{18}^2(\pi_{i,j}(\vec{t})) = \text{Equal}(\pi_{i,j}(\vec{t}))
\end{align*}
\]
One more atomic formula-type should take care of the one remaining predicate symbol, applicable only to sort-3 objects $u \in U_w^3$:

- $p_{i,j,k}^3$: “The first grid-location is closer to each of the other two grid-locations than they are to each other; and the three grid-locations all have the same column-number or the same row-number, or else the respective pairwise-differentials of their column-numbers and row-numbers are equal.”

In this case we have

$$\mathcal{M}_w, \bar{u} \models p_{i,j,k}^s \quad \text{iff} \quad S_w^* \models (p_{i,j,k}^s|\bar{a})_t^* [\bar{t} \leftarrow \bar{u}] \quad \text{iff} \quad S_w^* \models P_1^s(\pi_{i,j,k}(\bar{t})) [\bar{t} \leftarrow \bar{u}]$$

where it is straightforward to see, case by case, that $p_{i,j,k}^s$ describes $u$ just in case $u \in S_w^*(P_1^s)$, in the intended sense, for respective $1 \leq s \leq 3$. This may be verified because we know the intended meanings of the predicate symbols $P_1^s$ in $L_0^*$ when applied to blocks worlds $w$ (with blocks regarded as objects) and the respective formula-types $p_{i,j,k}^s$ in $L_0^*$ concerning the same blocks worlds $w$ (with tuples of blocks regarded now as worlds themselves). While $\mathcal{V}_w(p_{i,j,k}^s)$ may be said to specify the $\mathcal{M}_w$-character of the atomic formula-type $p_{i,j,k}^s$, it also determines the $S_w^*$-extension of the corresponding predicate symbol $P_{i,j,k}^s$ in $U_w^*$. The intended meanings of the various formula-types $p_{i,j,k}^s$ in $L_0^*$, as limited as they are, are thus able to determine correctly the intended extensions of each of the predicate symbols $P_{i,j,k}^s$ in $L_0^*$, for any given blocks world $w$. In this sense, this first version of $L_0$ is extensionally sufficient to interpret the first-order language $L_0^*$—and thus $L_{ipl}$, given the semantic isomorphism $\mu_\phi$ where $\mu_\phi^{-1}(S_w^*, V_w^*) = (S_w, V_w)$ (page 11).

On the other hand, one might aim not for mere extensional sufficiency but for descriptive adequacy, especially if the nature of material inference is at issue. The characterizations for the various worlds $u \in \mathcal{W}_w$ above are conceptually meager even if extensionally sufficient. If the aim is to interpret $L_{ipl}$ predicate symbols like ‘Cube’ or ‘Adjoins’ in ways that describe in some detail what such predicate symbols might actually mean, then mere extensional sufficiency would usually not be sufficient.

But what would be sufficient after all? Answers to that question may vary from one context to another; but each atomic formula-type $p_{i,j,k}^s$ in $L_0^*$
might in any case be packed with content, specifying in detail what it is to be a cube, or what it is for two blocks to adjoining one another. Formula-type $p_1^2$ might include an extensive description of cube-ness as a regular convex polyhedron, equal in detail to an encyclopedia entry if not an entire book [e.g., 49, 50].

Short of that, we might at least expect a more detailed “operationalizable” delineation of the various distinctive features of cubes, adjunctions, etc. In this case, the type of a single atomic formula $p_2^1|_i$ in $L_{0\square}$ that corresponds to the formulas $\text{Cube}(\pi_i(\vec{t})) (= (p_2^1|_i)_{\vec{t}}^*)$ in $L^*_{0\square}$ and thus should say something in detail about cube-ness might be a systematic collection of the following kinds of information:

- each face here $i$ has four vertices;
- each face here $i$ has four edges;
- each face here $i$ is a square;
- the angle between any two adjacent faces here $i$ is a right angle;
- the angle between two adjacent edges here $i$ is a right angle;
- the hull here $i$ has six faces;
- the hull here $i$ has eight vertices;
- the hull here $i$ has twelve edges;
- the color of the hull here $i$ is turquoise;
- the hull here $i$ encloses a volume equal to the cube of the length of the edge of any given face;
- the top here $i$ has four vertices;
- the top here $i$ is a square;
- etc.

As proper parts of single atomic formulas in $L_{0\square}$, we may refer to them as “features” of worlds. Each of these is easily viewed as involving a kind of measurement, and thus it is easy to see how each might be operationalized. At the same time, any one of these ways of characterizing a world is extensionally sufficient to pick out cubes in $LPL$ blocks worlds. With just the dozen or so possible features listed above (and one may list many more),
there are also the dozen-or-so-choose-two features constructed from pairs of these stating that the first holds true just in case the second one does. Etc. The conjunction of all such features in turn constitutes a rudimentary but complex operationally-grounded way of characterizing cube-like worlds.

Other features of cube-ness that are not distinctive of cubes as such but which will hold true in $L_{0□}$-worlds include things such as that

- the lengths of all edges here are equal
- the areas of all faces here are equal
- the volume enclosed here, by the hull is convex
- etc.

Such features will characterize any sort-1 world, cube-like or otherwise.

The point here is that such extensive information may be packaged into each one of the nineteen atomic formula-types in $L_{0□}$ that correspond one-to-one to respective monadic predicate symbols in $L^*_0$. The list above, which is distinctively characteristic of cube worlds, is easily enough modified to characterize tetrahedron worlds, larger-than worlds, left-of worlds, and so forth, to describe in detail what each one of the nineteen predicate symbols in $L^*_0$ is intended to mean.

Before going on, it may be instructive to return quickly to the example where $x, y$ is the $L^*_0$-formula

$$(P^1_1(\langle x \rangle) \land P^1_2(\langle y \rangle)) \rightarrow P^2_7(\langle x, y \rangle)$$

where, specifically, ‘$P^1_1$’ is Cube, and ‘$P^2_7$’ is Sameshape. Suppose that there is a structure $S^*_w$ for $L^*_0$ such that

$$S^*_w \models \psi_{x,y} [x\leftarrow u_1] [y\leftarrow u_2]$$

which we might otherwise write in the following way:

$$S^*_w \models \psi_{x,y} [\langle x, y \rangle\leftarrow \langle u_1, u_2 \rangle]$$

Namely, the formula $\psi_{x,y}$ is satisfiable by $\langle u_1, u_2 \rangle$ relative to structure $S^*$.

Is there an $L_{0□}$-formula $\varphi$ and an $L_{0□}$-model $\mathcal{M}_w$ such that its standard translation $\varphi^*_{(x,y)}$ is $\psi_{x,y}$? Apparently so. Let $\varphi$ be

$$(p^1_2 \land p^1_2) \rightarrow p^2_7|1,2$$

Then, the fact that $\psi_{x,y}$ is satisfiable by $\langle u_1, u_2 \rangle$ in a structure $S^*_w$ means that there is a blocks world $w$ with a number $n$ of blocks in it and there is an object $\langle u_1, u_2 \rangle \in U^*_w$ that satisfies the formula. Namely,
\[
S^*_w \models (P^1_2(\langle x \rangle) \land P^1_2(\langle y \rangle)) \ [(\langle x, y \rangle) \prec \langle u_1, u_2 \rangle]
\]
only if
\[
S^*_w \models P^2_7(\langle x, y \rangle) \ [(\langle x, y \rangle) \prec \langle u_1, u_2 \rangle]
\]
That is,
\[
S^*_w \models P^1_2(\langle x \rangle) [\langle x, y \rangle \prec \langle u_1, u_2 \rangle] \quad \text{and} \quad S^*_w \models P^1_2(\langle y \rangle) [\langle x, y \rangle \prec \langle u_1, u_2 \rangle]
\]
only if
\[
S^*_w \models P^2_7(\langle x, y \rangle) [\langle x, y \rangle \prec \langle u_1, u_2 \rangle]
\]
Now, each of the three \(L^*_0\)-formulas in this statement is the translation of an \(L_0\)-formula. Namely,
\[
S^*_w \models (p^1_2|_{\langle x, y \rangle}) [\langle x, y \rangle \prec \langle u_1, u_2 \rangle] \quad \text{and} \quad S^*_w \models (p^1_2|_{\langle x, y \rangle}) [\langle x, y \rangle \prec \langle u_1, u_2 \rangle]
\]
only if
\[
S^*_w \models (p^2_7|_{\langle x, y \rangle}) [\langle x, y \rangle \prec \langle u_1, u_2 \rangle]
\]
We have already established in turn that each of the three components of the latter statement is the case if and only if a corresponding statement concerning sentences being true at a world in the corresponding \(L_0\)-model \(M_w\) is also the case. Specifically, we have
\[
M_{w, \langle u_1, u_2 \rangle} \models p^1_2|_1 \quad \text{and} \quad M_{w, \langle u_1, u_2 \rangle} \models p^1_2|_2
\]
only if
\[
M_{w, \langle u_1, u_2 \rangle} \models p^2_7|_{1,2}
\]
That is,
\[
M_{w, \langle u_1, u_2 \rangle} \models (p^1_2|_1 \land p^1_2|_2) \to p^2_7|_{1,2}
\]
Or,
\[
M_{w, \langle u_1, u_2 \rangle} \models \psi
\]
We have thus shown that
\[
S^*_w \models \psi_{x,y} [\langle x, y \rangle \prec \langle u_1, u_2 \rangle] \quad \text{iff} \quad M_{w, \langle u_1, u_2 \rangle} \models \psi.
\]
It should be noted that respective versions of each of these two statements are true for any binary object/world \(\pi_{i,j}(\vec{t}) \in U^*_w\) because the two respective formulas are analytically or materially true with reference to any pair of blocks. Any two cubes have the same shape, as it were. We cannot prove either claim without appealing to the meanings of various “non-logical” components in either claim (in which case we have only pushed the enigma of “analytical truth” down a level). In any case, we in fact have
Given that $w$ is completely arbitrary here entails moreover that

\[ \models \forall (x, y) \, \psi_{x,y} \]

\[ \models \Box \varphi \]

The degree to which we may certify either one of these statements carries over to the other statement. This trickles up as well to the statement in $L_{tpd}$ that all cubes have the same shape.

2.5. The sentential language $L_0$

The next step here is to consider the language $L_0$ that results by eliminating the box operator from the language $L_{0\Box}$. This produces a fragment of $L_{0\Box}$ designated as follows:

\[ \varphi ::= p \mid \bot \mid \varphi_1 \rightarrow \varphi_2 \]

where $p \in \text{AtomicFormulas}_{L_{0\Box}}$. Recall that for any such atomic formula $p^*_l|\bar{a}$ in the particular language $L_{0\Box}$,

- $1 \leq l \leq 19$;
- $1 \leq s \leq 3$ such that
  - for $s = 1$, $1 \leq l \leq 6$,
  - for $s = 2$, $7 \leq l \leq 18$,
  - for $s = 3$, $l = 19$;
- $|\bar{a}| = s$ where $\bar{a} \in \mathbb{Z}^+ \cup \mathbb{Z}^+ \times \mathbb{Z}^+\cup \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$.

The nineteen atomic formula-types $p^*_l$ are matched one-to-one to predicate symbols in $L^*_{0\Box}$ in such a way as to be merely extensionally sufficient, maximally descriptively adequate, or something in between. For much the same reason that there are infinitely many atomic formulas in $L^*_{0\Box}$, there are infinitely many atomic formulas in $L_{0\Box}$ and thus in $L_0$. Namely, the subscript ‘$\bar{a}$’ used by the reference-fixing device ‘$\bar{a}$’ may be any 1-, 2-, or 3-tuple of the subscripts on sort-1 variables $x_i$ in $L^*_{0\Box}$.

Any given blocks world $w$ will have some fixed positive number of blocks $n_w$, up to a maximum of twelve. That does not limit the number of atomic formulas we may have insofar as any block may be referenced any finite
number of times in long-enough $L_0$-formulas. At the same time, any given $L_0$-formula will employ only a finite number of distinct reference-fixers ‘$\vec{a}$’.

Formulas in $L_0$ are properly termed “formulas” if we want to claim that they are not sentences (with specific contents) until they are used with reference to some fixed world. Following Kaplan [51, 52], Perry [48], and others, we may say that an atomic formula in $L_0$ (namely, a formula-type plus some appended reference-fixing device) expresses the character of an atomic sentence in $L_0$, whereas an atomic sentence qualifies as such because it is only with reference to a given world $\vec{u}$ that its reference-fixing device can fix the references of the various demonstrative and indexical elements in the respective formula. Only then does the formula acquire content and thus a truth-value at $\vec{u}$.

In fact, we probably need to distinguish four different steps in the construction of an $L_0$-sentence:

- formula-types $p_i^s$ for fixed $s, l$
- formulas $p_i^s|\vec{a}$ where $\vec{a}$ is a tuple of “variables” $i_1, i_2, \ldots$
- sentence-types $p_i^s|\vec{a}$ where $\vec{a}$ is tuple of positive integers
- sentences $p_i^s|\vec{a}$ in a context $\vec{u}$ where $\pi_{\vec{a}}(\vec{u})$ is determined

If we want a “sentential language” to be solely a language of sentences with definite contents, not just characters, then such a language with formula-types that utilize indexicals and demonstratives will be a sentential language only with reference to one or another blocks world $w$. There will be at least as many such languages as there are worlds $w$. This may or may not be a useful way to characterize languages, but it is perhaps not necessary if we adopt more or less the same attitude here toward yet-to-be-anchored formulas in $L_0$ that is taken toward open versus closed formulas $\varphi(x, \ldots)$ in first-order languages. Strictly speaking, a closed first-order formula (with no free variables) should not be termed a sentence if its components include indexicals or demonstratives. Yet there is much to be said for calling a first-order language of such formulas a “language.”

It is not clear what hinges on these concerns unless it is the case that a failure to make such distinctions would lead to debilitating ambiguities. In saying what an $L_0$-model is, the aim is to say in a systematic way how various formulas and/or sentence-types $p_i^s|\vec{a}$ acquire sentence-hood, so to speak, as well as to say, relative to worlds $\vec{u}$, what the truth-values of such sentences are. That is, because all formulas in $L_0$ are indexical, a model for such a language should provide the means for fixing referents of their indexical or demonstrative components.

A model $\mathcal{M}_w = \langle \mathcal{W}_w, \mathcal{V}_w \rangle$ for $L_0$ will simply be an $S5$ model $\mathcal{M}_w$ for $L_{0\Box}$ without the modal elements. Namely,
• \( \mathcal{W}_{0w} \) is \( U_{0w}^* = U_w^* = \mathcal{W}_w \), the set of all ordered \( n \)-tuples of sort-1 objects in \( U_{0w}^{1} = U_w^{1} \) (where any \( \vec{u} \in U_{0w}^* \) is thought of as a “world” of block-features);

• \( \mathcal{V}_{0w} : \text{AtomicFormulaTypes}_{L_0} \rightarrow 2^{\mathcal{W}_{0w}} \) is defined such that

\[
\mathcal{V}_{0w}(p^s) = \mathcal{V}_w(p^s) \subseteq U_{0w}^s \subseteq \mathcal{W}_{0w}^s.
\]

This essentially identifies how formulas and sentence-types with indexical or demonstrative terms might be anchored in a given domain of objects (each with worlds of features). We inductively define the notion of an \( L_0 \)-formula being \( \mathcal{M}_{0w} \)-true in a world \( \vec{u} = \langle u_1, \ldots, u_n \rangle \), for any \( n \in \mathbb{Z}^+ \), as follows:

- \( \mathcal{M}_{0w}, \vec{u} \models p^s |\vec{a} \iff \pi_{\vec{a}}(\vec{u}) \in \mathcal{V}_{0w}(p^s) \)
- \( \mathcal{M}_{0w}, \vec{u} \not\models \bot \)
- \( \mathcal{M}_{0w}, \vec{u} \models \phi \rightarrow \psi \iff \mathcal{M}_{0w}, \vec{u} \models \phi \) only if \( \mathcal{M}_{0w}, \vec{u} \models \psi \)

Again, an \( L_0 \)-model just is an \( \textbf{S5} \) \( L_0 \)-model in the literal sense that \( \mathcal{W}_{0w} = \mathcal{W}_w \) and \( \mathcal{V}_{0w} = \mathcal{V}_w \) (where the universal binary relation \( R_w \) adds nothing to the mix).

As with a model \( \mathcal{M}_w \), a model \( \mathcal{M}_{0w} \) serves to identify the \( w \)-extensions of predicate symbols in \( L_0^* \) and thus in \( L_{tpl} \). The various \( L_0 \)-formulas specify these extensions by stating which (measurable) features an \( s \)-tuple must have to be in the respective \( L_0^* \)-predicate-symbol’s extension.

More precisely, let

\[
\mathcal{X}_{0w} \overset{\text{def}}{=} \{ \langle \vec{u}, p^s \rangle \mid \vec{u} \in \mathcal{V}_{0w}(p^s) \}
\]

This is a unique relation in \( \mathcal{W}_{0w} \times \text{AtomicFormulaTypes}_{L_0} \) determined by \( \mathcal{V}_{0w} \). The set

\[
\mathcal{X}_{0w}(\vec{u}) = \{ p^s \mid \vec{u} \in \mathcal{V}_{0w}(p^s) \}
\]

is the set of all \( L_0 \)-formula-types \( p^s \) such that \( p^s |\vec{a}_s \) is \( \mathcal{M}_{0w} \)-true in \( \vec{u} \), where

- \( \vec{a}_s \) is \( \langle 1 \rangle \) if \( s = 1 \),
- \( \vec{a}_s \in \{ \langle 1, 2 \rangle, \langle 2, 1 \rangle \} \) if \( s = 2 \),
- \( \vec{a}_s \in \{ \langle 1, 2, 3 \rangle, \langle 1, 3, 2 \rangle, \langle 2, 1, 3 \rangle, \langle 2, 3, 1 \rangle, \langle 3, 1, 2 \rangle, \langle 3, 2, 1 \rangle \} \) if \( s = 3 \).

and

- \( \vec{a}_{\vec{u}} \) is \( \langle 1 \rangle \) if \( s = 1 = |\vec{u}| \),
\( \vec{a} \) is \( \langle 1, 2 \rangle \) if \( s = 2 = |\vec{u}| \),

\( \vec{a} \) is \( \langle 1, 2, 3 \rangle \) if \( s = 3 = |\vec{u}| \).

That is, in the latter case, \( \vec{u} \) is referenced as is in its given order: \( \vec{a} \) is uniquely determined once \( \vec{u} \) is fixed. Again, we may refer to the set \( X_{ow}(\vec{u}) \) as the \( \mathcal{M}_{ow} \)-characterization of \( \vec{u} \). Note that the domain of \( X_{ow} \) (which is the range of \( V_{ow} \)) is just \( U_{ow}^1 \cup U_{ow}^2 \cup U_{ow}^3 \); not the whole of \( V_{ow} \).

Let \( L_{0\vec{u}} \) be the fragment of \( L \) whose only atomic sentences \( A_{\vec{a}1\vec{a}} \) are \( p_j^s|_{\vec{a}} \) (as specified above) where \( |\vec{u}| = s \). There will be six such atomic sentences if \( s = 1 \), twenty-four if \( s = 2 \), and six if \( s = 3 \).

Then, \( X_{ow}(\vec{u}) \) specifies a row in a truth-table \( \mathfrak{T}_{0\vec{u}} \) for \( L_{0\vec{u}} \) whose reference columns are atomic \( L_{0\vec{u}} \)-sentences \( A_{\vec{a}1} \) (namely, \( p_j^s|_{\vec{a}} \)). The elements of \( X_{ow}(\vec{u}) \) correspond to atomic \( L_{0\vec{u}} \)-sentences that have the truth-value \( \text{true} \).

That is, \( X_{ow}(\vec{u}) \) determines a function assigning to each atomic sentence \( p_j^s|_{\vec{a}} \) a truth-value: \( \text{true} \) if \( p_j^s \in X_{ow}(\vec{a}) \) and \( \text{false} \) otherwise.

A truth-value assignment for a sentential language \( L \) is referred to as a \textit{world} in the usual construction of a basic modal language \( L \) from \( L \) [53, 397]. The single truth-value assignment \( X_{ow}(\vec{u}) \), on the other hand, determines a class of \( L_0 \)-worlds \( \vec{u} \) so far as the language \( L_0 \) is capable of characterizing such worlds.

For instance, let \( \langle u_1 \rangle \) and \( \langle u_2 \rangle \) be two distinct small cubes (sort-1 \( L_{0\vec{u}} \)-worlds) such that the first is to the left and in front of the second in a fixed blocks world \( w \). Then

\[
X_{ow}(\langle u_1 \rangle) = \{ p^1 \mid \langle u_1 \rangle \in V_{ow}(p^1) \} = \{ p^2, p^3 \}
\]

\[
X_{ow}(\langle u_2 \rangle) = \{ p^1 \mid \langle u_2 \rangle \in V_{ow}(p^1) \} = \{ p^2, p^3 \}
\]

These characterizations are identical and thus determine corresponding rows in \( \mathfrak{T}_{0(u_1)} \) and \( \mathfrak{T}_{0(u_2)} \). They are not able as such to distinguish \( \langle u_1 \rangle \) and \( \langle u_2 \rangle \). This one characterization rather determines a class of sort-1 worlds (namely, those that are small cubes). On the other hand,

\[
X_{ow}(\langle u_1, u_2 \rangle) = \{ p^2 \mid \langle u_1, u_2 \rangle \in V_{ow}(p^2) \} = \{ p^5, p^6, p^7, p^8 \}
\]

is a characterization of the \textit{pair} \( \langle u_1, u_2 \rangle \) (in a fixed order). This characterization distinguishes \( u_1 \) and \( u_2 \) by way of different asymmetric spatial relations, viz., asymmetric spatial \textit{properties} of the pair. For example, \( p^2_{11,1,2} \) is \( \mathcal{M}_{ow} \)-true while \( p^2_{11,2,1} \) is \( \mathcal{M}_{ow} \)-false. That is, \( \langle u_1, u_2 \rangle \in V_{ow}(p^2_{11}) \) whereas \( \langle u_2, u_1 \rangle \notin V_{ow}(p^2_{11}) \). Nevertheless, this will be the case in any blocks world with two small cubes where the first is to the left and in front of the second—and there are many such blocks worlds. We may be able to distinguish sort-1
$L_{0\Xi}$-worlds by means of facts in sort-2 $L_{0\Xi}$-worlds, but distinct sort-2 $L_{0\Xi}$-worlds may have identical sort-2 characterizations. Etc.

Then, consider the language $L_{0w}$ where

- $\text{AtomicFormulas}_{L_{0w}} = \bigcup_{\vec{a}} \text{AtomicFormulas}_{L_{0\vec{a}}}$
  \[ = \bigcup_{\vec{a}} \{ p^*_1 | \vec{u} \in \mathcal{V}_{0w}(p^*) \} \]
- $X_{0w} = \bigcup_{\vec{a}} \mathcal{X}_{0w}(\vec{u}) = \bigcup_{\vec{a}} \{ p^* | \vec{u} \in \mathcal{V}_{0w}(p^*) \}$

The latter characterizes each sort-1, sort-2, and sort-3 world $\vec{u}$ in $w$ according to specifications stipulated by $M_{0w}$.

In effect, $X_{0w}$ specifies a row in a larger “truth-table” $\mathcal{X}_{0w}$ whose reference columns are atomic $L_0$-sentences whose contents are that

“$p^*|_{\vec{a}}$ is the case with respect to $\vec{u}$” for each choice of appropriate values for $s$, $l$, $\vec{a}$, and $\vec{u}$. The numbers of possible values for $s$ and $l$ are finite as are those for $\vec{a}$ and $\vec{u}$, given that we are considering only worlds $\vec{u}$ in the domain of $X_{0w}$. The resulting set of atomic $L_{0w}$-sentences is thus a finite set \{ $A_{w1}, A_{w2}, A_{w3}, \ldots$ \}.

The elements of $X_{0w}$, then, are those atomic $L_{0w}$-sentences that have the truth-value TRUE in $\mathcal{X}_{0\vec{u}}$ for some $\vec{u}$ in the range of $\mathcal{V}_{0w}$. Having fixed $w$, and given that the various worlds $\vec{u}$ are distinct, the various truth-tables $\mathcal{X}_{0\vec{u}}$ have disjoint sets of reference columns and thus on purely formal grounds are mutually consistent.

The set $X_{0w}$ thus determines a single function (a single row in $\mathcal{X}_{0w}$) assigning to each atomic sentence $A_{wi}$ a truth-value: TRUE if $A_{wi} \in X_{0w}$ and FALSE otherwise.

Except for being anchored in a specific concrete blocks-world $w$, the truth-value assignment $X_{0w}$ would determine not a single blocks world but a class of blocks worlds (that is, the world is underdetermined by the data). Namely, any given blocks world $w$ will correspond to exactly one row $X_{0w}$ in the set $\mathcal{X}_{0w}$ of all such truth-value assignments to the set of atomic $L_{0w}$-sentences. On the other hand, there may be an “isomorphic” but distinct blocks world $w'$ (matched block for block with $w$ but with certain “minor” differences, say, in the locations of respective blocks). Then, $X_{0w'}(\vec{u}')$ could match $X_{0w}(\vec{u})$ sentence for sentence so as to be indistinguishable if it was not known beforehand which world was $w$ and which was $w'$. In this sense, $X_{0w}$ uniquely characterizes a blocks worlds only because the given blocks world is given beforehand to fix references.

Consider, for instance, a class of blocks worlds $w$ each with two blocks in it—a cube and a tetrahedron, say—placed anywhere on the grid but so as to
preserve key spatial relations. Each such \(w\) will be described by the “true” sentences in exactly one row of \(\mathfrak{S}_{0w}\) given that those sentences (by design) are able to determine the positions of blocks relative only to one another, not relative to the grid.

In general, consider the truth-table \(\mathfrak{S}_{0w}\) for sentence-types determined as above by a given blocks world \(w\). The set \(\mathfrak{X}_0\) will determine a single row in this table (and serve to determine a class of similar worlds \(w'\)), but the other rows represent other classes of possible blocks worlds where \(w\) and \(L_{0w}\) will have served to specify a finite set of distinctive atomic sentence-types with respective reference columns in \(\mathfrak{S}_{0w}\).

In many cases, the class of “possible” blocks worlds corresponding to a given row will be empty. There will be no such blocks worlds corresponding to rows where the material contents of the various atomic sentences are mutually incompatible. For example, there can be no world (under intended construals of all the terms involved) in which a given block is both small and large. Thus there will be no world \(w\) corresponding to any row (and there are a lot of them) where two sentences with the following contents are assigned the value TRUE:

\[
\text{“} p_i^{1}|_1 \text{ is the case with respect to } \langle u_1 \rangle \text{”}
\]

\[
\text{“} p_j^{1}|_1 \text{ is the case with respect to } \langle u_1 \rangle \text{”}
\]

As one works through the list \(p_1^1, \ldots, p_{19}^2\) of formula-types to consider what they mean, it becomes obvious that a great many rows in \(\mathfrak{S}_{0w}\) will determine the empty set of blocks worlds.

Let \(\mathfrak{S}_{0w}^{lpl}\) be the portion of \(\mathfrak{S}_{0w}\) none of whose rows determine the empty set of blocks worlds. This will depend directly on the characters of \(L_0\)-formula-types. Then we have the following definitions:

- An \(L_0\)-sentence is a tautology iff, for any blocks world \(w\), it is true in every row in \(\mathfrak{S}_{0w}\).
- An \(L_0\)-sentence is an analytic or material truth iff, for any blocks world \(w\), it is true in every row in \(\mathfrak{S}_{0w}^{lpl}\).

Any tautology is of course analytically true; but not vice versa. A simple example of a tautology would be any sentence \(A_j \rightarrow A_j\), whereas a material truth will be, for example, \(A_j \leftrightarrow A_k\) (clearly not tautologous if \(j \neq k\)) where

\(A_j\) is “\(p_j^2|_{1,2}\) is the case with respect to \(\langle u_1, u_2 \rangle\)”

\(A_k\) is “\(p_k^2|_{2,1}\) is the case with respect to \(\langle u_1, u_2 \rangle\)”

31
Namely, a particular block is to the left of another block just in case the latter is to the right of the former.

So recall the original thesis that we may interpret predicate symbols in $L_{lpl}$ as “classes of models,” versus a standard extensional interpretation. The thesis is that extensional interpretations are not particularly “explanatory” whereas interpretations of the former kind are.

Consider, then, the class of all models $\mathfrak{M}_{0w}$, for any blocks-world $w$, such that there is a feature-world $\hat{u} \in U^l_{0w}$ (namely, a sort-1 feature-world) such that for any feature-world $\vec{u} \in W_{0w}$ and for any $i \in \mathbb{Z}^+$, if $\hat{u} = \pi_i(\vec{u})$, then $\mathfrak{M}_{0w}, \vec{u} \models p^i_2$. This is the set of all $L_0$-models, for any $w$, with the single common feature that there is at least one sort-1 world-of-features $\hat{u}$ such that all of the faces in the feature-world $\hat{u}$ have four edges, four vertices, etc.

The latter class corresponds to the class of all rows in $\mathcal{T}^{lpl}_{0w}$, for any blocks-world $w$, where there is an $i \in \mathbb{Z}^+$ such that $p^i_1$ is true.

The common (invariant) feature of all such rows in $\mathcal{T}^{lpl}_{0w}$ and/or models $\mathfrak{M}_{0w}$ is that there is a block in $w$ such that all of its faces have four edges.

The significant point here is that an interpretation of, for instance, the predicate symbol “Cube” in $L_{lpl}$ may in this way be achieved by reference not to all of the things in its extension but rather to the features an object must have to be a cube. The languages $L_{lpl}$ and $L_0$ are languages for talking about features of blocks, after all, rather than about blocks as such. It is significant here that $p^i_1$-sentences and $p^i_3$-sentences will always be false in such rows in $\mathcal{T}^{lpl}_{0w}$ where respective $p^i_2$-sentences are true—not by mere stipulation but because if the faces in a world of features all have four edges, then they simply cannot have some other number of edges. This would allegedly explain why it is (and not merely state in equivalent terms) that if something in a blocks world $w$ is a cube, then it is neither a tetrahedron nor a dodecahedron.

3. Some Results

We have looked at four systematically-related languages where analytical truths in the one correspond to analytical truths (with compatible informational contents) in the others. The enigmatic nature of analytic or material truth (analytic or material consequence, analytic or material equivalence, material inference, etc.) is not in this way explained but merely takes different forms from one language to the next. To ground the dictionary-like circularity of this process might require some language at some level that preserves this correspondence of analytic truths but which also is such that truths in that language (analytic or not) are in themselves immediately obviously so. Analytic truths in $L_{lpl}$ would then be explainable by tracing their correspondence to the immediately obvious truths in the latter language. At
that point, we will have achieved a kind of semantic clarity, both formal and material.

Obviously Peirce [7] was not thinking in exactly these terms, but his statement of the pragmatic maxim would seem to entail that such perfect clarity is not soon to be expected. Rather, the best we can expect to do is to operationalize our formulas and predicate symbols so that what we mean when using such elements of language can eventually be cashed out explicitly in operational terms such that alleged truths are certified by means of predictable sensible effects of specific practices. This presents a promising perspective on analytical truth (etc.) insofar as such practices might reach beyond mere talk even if they might not guarantee perfect clarity. Of course, we cannot reach beyond our own reach, but our reach is not limited to what is achievable by linguistic discourse alone. As important as linguistic discourse may be to understanding human nature (and especially our nature as thinking things), our abilities go beyond abilities to use “vocabularies” to include abilities to use our senses, our bodies, tools, measuring devices, etc. Operationalizing our discourse in terms of the latter kinds of abilities would appear to be what we have to do ultimately to achieve solid if not final clarity that reaches outside of language as such.

It has been noted occasionally in the presentation of the language $L_{0\Box}$ that its atomic formulas are easily operationalized—which is to say that, on the surface, they are relatively easily operationalized in terms of abilities to use our senses, measuring devices, etc., whereas atomic sentences in the languages $L_{lpl}$ and $L_{0\Box}^*$ employ predicate symbols whose meanings are supposed to be not so simply connected to, say, “sensory effects.” This illustrates the granularity of information where $L_{lpl}$ and $L_{0\Box}^*$ are more coarsely grained than $L_{0\Box}$ and $L_0$. We might also think of $L_{lpl}$ and $L_{0\Box}^*$ as analogues of more abstract theoretical languages. The move from $L_{0\Box}^*$ to $L_{0\Box}$, on the other hand, is not merely formal but introduces more detailed content in a way that moves closer to what in the semantic conception of scientific theories [34, 35] count as languages and thus models of data. The work presented here explores not only how to make that move but how to make it in such a way that the resulting languages and models are oriented not just to data but also to the operations by which such data are obtained.

The move from $L_{0\Box}^*$ to $L_{0\Box}$ is substantial in this regard in that atomic $L_{0\Box}$-formulas are cast in operationalizable terms of counting and measuring. This move is not merely formal but material. The move from $L_{0\Box}$ to $L_0$ was again largely formal, though not for nothing. The formal move from $L_{lpl}$ to $L_{0\Box}^*$ was needed to make the more substantial move from $L_{0\Box}^*$ to $L_{0\Box}$. The move to $L_0$ should make it easier to link the present discussion to a treatment of ability-based languages, models and logics that explicitly include references
to executions of abilities that reach into realities beyond mere discourse.

Appendix A. The first-order blocks language \( L_{lpl} \)

The language \( L_{lpl} \), the “blocks language” designed to talk about “blocks worlds,” is used in [33] (\( LPL \)) to teach elementary first-order logic. The particular concepts we want to “clarify” will be (only) the nineteen concepts associated respectively with the nineteen predicate symbols employed by \( L_{lpl} \), grouped in Table A.2 according to their arities.

These symbols pretty much wear their intended meanings on their sleeves, though some explanation is required given that we want to disallow vagueness and ambiguity in their application to suitable objects in appropriate domains. Their meanings in some cases are not quite so straightforward as one might think.

Specifically, suitable objects will be objects in blocks worlds as depicted by the software application Tarski’s World (version 6.7.1 as of this writing). A screenshot of a sample world is accessible at

\( \langle \text{http://ggww2.stanford.edu/GUS/SHARED/images/Tarski6.jpg} \rangle \)

Each predicate symbol in the blocks language has a single, fixed “arity” (roughly, the number of name tokens needed to form an atomic sentence); and it is interpreted by a determinate property or relation of the same arity as the predicate symbol. This means that, given any property and object, “there is a definite fact of the matter whether or not the object has the property,” or given any relation and any appropriate number of objects in a particular order, there is a definite fact of the matter whether or not the objects in that order stand in the relation. This latter kind of claim is what we want to render more precisely in pragmatist terms—not to require such determinacy across the board but simply to say more precisely what it amounts to.

We want to preserve the overall aim in \( LPL \), which is to construct a particular first-order language—a blocks language—designed to illustrate a
great many interesting features of first-order languages. English interpretations for typical atomic sentences in this first-order blocks language are given in Table A.3. Note, for example, that in the world depicted in the screenshot referenced above, no two blocks adjoin one another. Two blocks must be next to each other in the same row or column for this relation to hold. Similarly, in regard to the intended sense of the Between predicate, the medium cube is between the other two cubes, and the small dodecahedron is between the two medium tetrahedra, whereas, while the leftmost small tetrahedron is collinear with the small dodecahedron and the rightmost medium tetrahedron, it is not “between” them in the sense intended here. The latter relation holds only among three blocks that are in the same row, same column, or same 45-degree diagonal.

Note, too, that there are a number of facts about blocks and blocks worlds that are not expressible in the blocks language we are considering with just the nineteen predicate symbols listed above. As just noted, collinearity by itself is a determinate relation which the blocks language does not accommo-
Blocks are colored—in fact, all blocks of a given shape have the same color—but no predicate symbol is available to state such facts. No predicate symbols are provided to mention actual positions on the board; rather only relative positions may be described though, again, with no means to describe relative distances. The list goes on.

Such limitations taken together actually constitute a potentially informative feature of the blocks language rather than a failing. In particular, these limitations are in line with the fact (no doubt) that natural languages have analogous semantic blind spots reflecting historical and cultural biases and experiential perspectives more generally.

The key question here concerns how to “clarify” the concepts associated with the various predicate symbols in the blocks language \( L_{lpl} \) in such a way that analytic consequence and material inference are not so mysterious. There are a few options worth mentioning for the sake of contrast with pragmatist methods.

One way that may seem promising is the so-called axiomatic method, introducing “meaning postulates,” utilizing the very blocks language \( L_{lpl} \) that we are trying to elucidate, “by systematically expressing facts about the predicates involved in our inferences” [33, 284]. The inadequacy of this method is discussed at some length in [33, 283–288, 338–341] in the context of linking a relatively broad intuitive notion of consequence with the more restricted notion of first-order consequence. The topic of logical consequence nicely highlights the problem of meaning by illustrating the apparently principled workings of the latter in deductive inference even though we do not seem to understand fully the nature of whatever principles seem to be at work there. For example, the sentence \( \text{Cube}(c) \) is a logical consequence of the premise \( \forall x(\text{Cube}(x) \leftrightarrow \text{SameShape}(x, c)) \) not solely on first-order grounds but also (necessarily) by virtue of the meanings of the predicate symbols—illuminating an analytic or material consequence that is not a first-order consequence.

Second, it might serve to expand the blocks language \( L_{lpl} \) to allow more and more detailed expressibility, particularly in conjunction with using the axiomatic method. Nevertheless we will simply end up with the same problem involving other predicate symbols if not those of the current blocks language. We will not be able to plug all of the semantic holes just by adding more and more interpreted predicate symbols. On the contrary, the obvious limitations of the blocks language as given in \( LPL \) serve to keep the problem in view rather than hide it behind more complex refinements that obscure the problem but do not solve it.

Third, we might bring to bear a separate language—say, a separate language of set theory, or of geometry, or of physics—to be used as nature’s “language” (metaphorically speaking) of properties, relations, things, facts,
states of affairs, and such, whereas the blocks language $L_{bld}$ exemplifies our human language (literally) for formulating talk about such things. In what is supposed to be an ontologically stripped-down form, this is essentially the method employed in specifying standard extensional structures for first-order languages. Clearly, this only puts off the problem of meaning, substituting in its place purely extensional “interpretations” of predicate symbols. Whatever one thinks of the extensionalist strategy, results such as the Löwenheim-Skolem Theorem and the existence of nonstandard set-theoretic models of Peano arithmetic establish once and for all that first-order languages pertaining to almost any nontrivial subject-matter typically are “not rich enough to be able to capture various concepts that we implicitly assume when thinking about the intended universe” of the given subject matter [33, 546–550]. One is thus inclined to conclude that this third option will not work either.

Appendix B. Grid locations

It will be convenient to clarify the way that board positions are represented—namely, to use sixty-four different single numerals $11, 21, \ldots, 78, 88$ where the eight columns are represented by respective tens digits in these numerals while the eight rows are represented by the eight units digits. This scheme is depicted in Table B.4.

To be able to refer separately to the column or row of a given position represented by numeral $x$, we define $\text{col}(x) =$ the digit in the tens’ place in $x =$ the tens digit in $x$. Likewise $\text{row}(x) =$ the digit in the ones’ place in $x =$ the units digit in $x$. 

Table B.4: Numerals representing board positions

<table>
<thead>
<tr>
<th>row</th>
<th>11</th>
<th>21</th>
<th>31</th>
<th>41</th>
<th>51</th>
<th>61</th>
<th>71</th>
<th>81</th>
</tr>
</thead>
<tbody>
<tr>
<td>col</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>28</td>
<td>38</td>
<td>48</td>
<td>58</td>
<td>68</td>
<td>78</td>
<td>88</td>
</tr>
<tr>
<td></td>
<td>17</td>
<td>27</td>
<td>37</td>
<td>47</td>
<td>57</td>
<td>67</td>
<td>77</td>
<td>87</td>
</tr>
<tr>
<td>(left)</td>
<td>16</td>
<td>26</td>
<td>36</td>
<td>46</td>
<td>56</td>
<td>66</td>
<td>76</td>
<td>86</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>25</td>
<td>35</td>
<td>45</td>
<td>55</td>
<td>65</td>
<td>75</td>
<td>85</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>24</td>
<td>34</td>
<td>44</td>
<td>54</td>
<td>64</td>
<td>74</td>
<td>84</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>23</td>
<td>33</td>
<td>43</td>
<td>53</td>
<td>63</td>
<td>73</td>
<td>83</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>22</td>
<td>32</td>
<td>42</td>
<td>52</td>
<td>62</td>
<td>72</td>
<td>82</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>21</td>
<td>31</td>
<td>41</td>
<td>51</td>
<td>61</td>
<td>71</td>
<td>81</td>
</tr>
</tbody>
</table>

(180x612) (left)
References


